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On the regularity of the L_p Minkowski problem ☆

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ABSTRACT

The L_p Minkowski problem is equivalent to solve the Monge–Ampère equation

$$\det(u_{ij} + u\delta_{ij}) = u^{p-1} f, \quad \text{on } \mathbb{S}^n.$$

Since it is degenerate for $1 < p < n+1$, the equation has no smooth solution even when the prescribed positive function f is smooth. In this paper, the C^∞ regularity for the solution is obtained for $2 < p < n+1$ by adding a gradient condition on f.

1. Introduction and main results

For a given continuous function $f : \mathbb{S}^n \to \mathbb{R}$, *the L_p Minkowski problem* seeks the solution $u : \mathbb{S}^n \to [0, \infty)$ to the Monge–Ampère equation

$$\det(u_{ij} + u\delta_{ij}) = u^{p-1} f \quad \text{on } \mathbb{S}^n, \tag{1.1}$$

where u_{ij} is the covariant derivative of u with respect to an orthonormal frame on $\mathbb{S}^n$. The case $p = 1$ is classical with landmark contributions by Minkowski, Aleksandrov, Fenchel and Jessen, Lewy (see e.g. [30–32]), Nirenberg [29], Calabi [8], Cheng and Yau [10], Caffarelli [6,7] and others.

Eq. (1.1) is degenerate for $1 < p < n+1$, even when f is positive and smooth. Guan and Lin [17] and Hug et al. [20] constructed an example which showed that the boundary of the convex body may

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touch the origin, hence u does not have a positive lower bound. The main aim of this paper is to find the condition on f, which guarantees that the support function of the convex body is positive and then Eq. (1.1) has a smooth solution. Finding smooth solutions is a very interesting problem for the degenerate Monge–Ampère equations (see [19] and its references). The key is how to improve the solutions of degenerate Monge–Ampère equations from $C^{1,1}$ to C^{∞}. Even for $C^{1,1}$ estimates, it also needs the condition $\frac{n+1}{2} < p < n+1$, e.g. see Chou and Wang [11] and Guan and Lin [17].

A central open problem within the L_p Brunn–Minkowski theory is the existence and uniqueness of the solution to the L_p Minkowski problem for general data and general $p \in \mathbb{R}$. Lutwak [23] showed that there is a corresponding L_p Brunn–Minkowski theory for $p \geqslant 1$. Moreover he obtained the existences and uniqueness of weak solutions for generalized L_p Minkowski problem with prescribed positive even functions. Later, the regularity of the solutions in this case was established by Lutwak and Oliker [25].

For a convex body K, the support function of K is defined by $u_K = \max_{y \in K} \langle x, y\rangle : \mathbb{R}^{n+1} \to \mathbb{R}$. The surface area measure, $S(K, \cdot)$ of the convex body K is a Borel measure on $\mathbb{S}^n$ such that

$$\lim_{\varepsilon \to 0^+} \frac{V(K+\varepsilon Q) - V(K)}{\varepsilon} = \int_{\mathbb{S}^n} u_Q\, S(K, dx), \quad \text{for each convex body } Q, \tag{1.2}$$

where $V(K)$ denotes $(n+1)$-dimensional volume of a convex body K in $\mathbb{R}^{n+1}$, and $K + \varepsilon Q$ is the Minkowski combination defined by

$$u(K+\varepsilon Q, \cdot) = u(K, \cdot) + \varepsilon u(Q, \cdot).$$

Suppose $p > 1$ is fixed and K is a convex body that contains the origin in its interior. The L_p surface area measure, $S_p(K, \cdot)$ of the convex body K is a Borel measure on $\mathbb{S}^n$ such that

$$\lim_{\varepsilon \to 0^+} \frac{V(K+_p \varepsilon Q) - V(K)}{\varepsilon} = \frac{1}{p} \int_{\mathbb{S}^n} u_Q^p\, S_p(K, dx), \tag{1.3}$$

for each convex body Q that contains the origin in its interior. Here $K +_p \varepsilon Q$ is the Minkowski–Firey L_p combination (see [12,24,32]) defined by

$$u(K +_p \varepsilon Q, \cdot)^p = u(K, \cdot)^p + \varepsilon u(Q, \cdot)^p.$$

Lutwak [23] established the existence of the L_p surface area measure and showed that

$$S_p(K, \cdot) = u_k^{1-p} S(K, \cdot). \tag{1.4}$$

The general L_p Minkowski problem asks: Given a real p, what are the necessary and sufficient conditions on a Borel measure μ on the unit sphere, $\mathbb{S}^n$, such that there exists a convex body K in $\mathbb{R}^{n+1}$ with support function u_K and surface area measure S_K so that

$$u_K^{1-p}\, dS_K = d\mu. \tag{1.5}$$

Lutwak, Yang, Zhang, and their co-authors [18,20,28] obtained important results on the L_p Minkowski problem for polytopes and applications of the L_p Minkowski problem to sharp affine invariant L_p Sobolev inequalities [26,27].

From the view of partial differential equation theory, Guan and Lin [17] and Chou and Wang [11] focused on the existence and regularity of the L_p Minkowski problem. A solution for $p > n+1$ was given by Guan and Lin [17] and independently by Chou and Wang [11]. The work of Chou and Wang

[11] goes further and solves the problem for $p \in (-n-1, 1)$. In addition, it discusses a Minkowski problem in centroaffine geometry, see also [1,2].

Now we state our main result.

Theorem 1.1. *Let $0 < f \in C^\infty(\mathbb{S}^n)$ and $2 < p < n+1$. If*

$$\max_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} \log f| \leqslant \frac{p-2}{\pi}, \tag{1.6}$$

then the L_p Minkowski problem has a positive smooth solution.

Remark 1.2. If f is constant, Theorem 1.1 reproves the result of Lutwak and Oliker [25], which says that for any given even function $0 < f \in C^\infty(\mathbb{S}^n)$, and $p > 1$, the L_p Minkowski problem has a positive C^∞ solution. On the other hand, Chou and Wang [11] and Guan and Lin [17] obtained $C^{1,1}$ regularity of solutions to (1.1) for $\frac{n+1}{2} < p < n+1$ and $0 < f \in C^{1,1}(\mathbb{S}^n)$, which may imply that our condition (1.6) is not limited.

The paper is organized as follows: the key gradient type estimates implying the upper bound of u are presented in Section 2. We will obtain the lower bound of u, and then we will prove Theorem 1.1 in Section 3. Lastly, we prove the a priori estimate in Theorem 4.1 for $p < 2$ if the solution is even in Section 4.

2. A gradient estimates implying an upper bound of u

In this section, we obtain a gradient type estimate which implies the upper bound of u. These results are taken from [22]. Moreover it holds for more general $0 < p < n+1$. In the sequel, we use ∇ to represent the gradient on $\mathbb{S}^n$ instead of $\nabla_{\mathbb{S}^n}$.

Lemma 2.1. *Assume that $u \in C^3(\mathbb{S}^n)$ is a solution of (1.1) for $0 < p < n+1$, then there is a positive constant $\alpha \in (0, \frac{p}{n+2})$ such that*

$$\frac{|\nabla u|}{u^\alpha} \leqslant C_1 \tag{2.1}$$

where C_1 depends only on $\|f\|_{C^1(\mathbb{S}^n)}$, $\min_{\mathbb{S}^n} f$ and $\max_{\mathbb{S}^n} f$, α, n, p.

Proof. Suppose that $G = \frac{|\nabla u|}{u^\alpha}$ attains its maximum at the point $x_0 \in \mathbb{S}^n$. Without loss of generality, we choose normal coordinates at x_0 such that

$$u_1 = |\nabla u|, \tag{2.1a}$$

$$u_i = 0 \quad \text{for } i \geqslant 2. \tag{2.1b}$$

Writing Eq. (1.1) as

$$\log\det(u_{ij} + u\delta_{ij}) = \log\det(w_{ij}) = (p-1)\log u + \log f,$$

and differentiating in direction $e_1 \in T_{x_0}\mathbb{S}^n$, we obtain

$$w^{ij} w_{ij1} = \frac{f_1}{f} + (p-1)\frac{u_1}{u}.$$

At x_0, we have

$$0 = (\log G)_i = \frac{u_{1i}}{u_1} - \alpha \frac{u_i}{u},$$

which implies

$$u_{11} = \alpha \frac{u_1^2}{u}, \qquad u_{1i} = 0 \quad \text{for } i \geqslant 2. \tag{2.2}$$

We may as well assume that $[u_{ij}(x_0)]$ is diagonal. Hence, we have the following inequality at x_0 (we note that x_0 is also the maximal point of $\log G$),

$$\begin{aligned}
0 \geqslant w^{ij}(\log G)_{ij} &\geqslant w^{ij}\left(\frac{u_{1ij}}{u_1} - \frac{u_{1i}u_{1j}}{u_1^2} - \frac{\alpha u_{ij}}{u} + \frac{\alpha u_i u_j}{u^2}\right) \\
&= \frac{1}{u_1} w^{ij} w_{ij1} - w^{11} - \frac{1}{u_1^2} w^{11} u_{11}^2 - \frac{\alpha}{u} w^{ij} u_{ij} + \frac{\alpha}{u^2} w^{11} u_1^2 \\
&= \frac{1}{u_1} w^{ij} w_{ij1} - w^{11} - \frac{\alpha}{u} w^{11} u_{11} - \frac{\alpha n}{u} + \alpha \sum_i w^{ii} + \frac{\alpha}{u^2} w^{11} u_1^2 \\
&= \frac{1}{u_1} w^{ij} w_{ij1} - w^{11} + \left(\alpha - \alpha^2\right) w^{11} \frac{u_1^2}{u^2} - \frac{\alpha n}{u} + \alpha \sum_i w^{ii},
\end{aligned} \tag{2.3}$$

where we have used $u_{1ij} = w_{1ij} - \delta_{1i} u_j = w_{ij1} - \delta_{1i} u_j$ in the last equality.

Note that

$$w^{11} = \frac{1}{u_{11} + u} = \frac{u}{\alpha u_1^2 + u^2},$$
$$w^{ij} w_{ij1} = \frac{(p-1)u_1}{u} + \frac{f_1}{f}.$$

Thus

$$\begin{aligned}
0 &\geqslant \frac{1}{u_1} w^{ij} w_{ij1} - w^{11} + \left(\alpha - \alpha^2\right) w^{11} \frac{u_1^2}{u^2} - \frac{\alpha n}{u} + \alpha \sum_i w^{ii} \\
&= \frac{p-1}{u} + \frac{f_1}{f u_1} - \frac{u}{\alpha u_1^2 + u^2} + \left(\alpha - \alpha^2\right) \frac{u_1^2}{\alpha u u_1^2 + u^3} - \frac{\alpha n}{u} + \alpha \sum_i w^{ii}.
\end{aligned} \tag{2.4}$$

We divide it into two cases to deal with the inequality according to the value of $u(x_0)$. There exists a positive constant M depending only on p, n, f and to be determined such that

- Case 1: $u(x_0) \leqslant M$;
- Case 2: $u(x_0) \geqslant M$.

Case 1: $u(x_0) \leqslant M$: Subcase (1_A): $u_1 \leqslant \frac{u}{\alpha}$, then $\frac{u_1}{u^\alpha} \leqslant \frac{u^{1-\alpha}}{\alpha} \leqslant \frac{M^{1-\alpha}}{\alpha}$.

Subcase (1_B): $u_1 \geqslant \frac{u}{\alpha}$, then $\frac{u^2}{\alpha u_1^2 + u^2} = \frac{1}{\alpha(\frac{u_1}{u})^2 + 1} \leqslant \frac{\alpha}{1+\alpha} < \alpha$.

Combining this and (2.4) we have:

$$\begin{aligned}0 &\geqslant \frac{p-1}{u}+\frac{f_1}{fu_1}-\frac{u}{\alpha u_1^2+u^2}+\left(\alpha-\alpha^2\right)\frac{u_1^2}{\alpha uu_1^2+u^3}-\frac{\alpha n}{u}+\alpha\sum_i w^{ii}\\ &=\frac{p-1}{u}+\frac{f_1}{fu_1}+\frac{1}{u}\left[1-\alpha-\frac{(2-\alpha)u^2}{\alpha u_1^2+u^2}\right]-\frac{\alpha n}{u}+\alpha\sum_i w^{ii}\\ &\geqslant \frac{1}{u}\left[p-(3+n)\alpha+\alpha^2\right]+\frac{f_1}{fu_1}.\end{aligned}\tag{2.5}$$

Here we set $p \geqslant (n+3)\alpha$, then it yields

$$0 \geqslant \frac{\alpha^2}{u}-\frac{|\nabla f|}{fu_1},$$

so

$$\frac{u_1}{u^\alpha} \leqslant \frac{|\nabla f|}{\alpha^2 f}M^{1-\alpha}.\tag{2.6}$$

Case 2: $u(x_0) \geqslant M$.

Subcase (2_A): $u_1 \geqslant C_3 u^{\frac{p-1}{n}}$, where C_3 is chosen such that

$$\frac{1}{2}\alpha n C_3(\max f)^{-\frac{1}{n}}-\max\frac{|\nabla f|}{f}>0.\tag{2.7}$$

Furthermore we choose M large enough such that

$$\frac{1}{2}\alpha n M^{1-\frac{p-1}{n}}(\max f)^{-\frac{1}{n}}>2.\tag{2.8}$$

From (2.4), we get

$$\begin{aligned}0 &\geqslant \frac{p-1}{u}+\frac{f_1}{fu_1}-\frac{u}{\alpha u_1^2+u^2}+\left(\alpha-\alpha^2\right)\frac{u_1^2}{\alpha uu_1^2+u^3}-\frac{\alpha n}{u}+\alpha\sum_i w^{ii}\\ &=\frac{p-1}{u}+\frac{f_1}{fu_1}+\frac{1-\alpha}{u}-(2-\alpha)\frac{u^2}{\alpha uu_1^2+u^3}-\frac{\alpha n}{u}+\alpha\sum_i w^{ii}\\ &\geqslant \frac{p-(1+n)\alpha}{u}+\left(\frac{f_1}{fu_1}+\frac{\alpha}{2}\sum_i w^{ii}\right)+\left(\frac{\alpha}{2}\sum_i w^{ii}-\frac{2}{u}\right),\end{aligned}$$

which leads to a contradiction provided that

$$0 \leqslant \left(\frac{f_1}{fu_1}+\frac{\alpha}{2}\sum_i w^{ii}\right)+\left(\frac{\alpha}{2}\sum_i w^{ii}-\frac{2}{u}\right).\tag{2.9}$$

Indeed, (2.2) tells us that $w_{ij}=u_{ij}+u\delta_{ij}$ may be diagonal at x_0, and then

$$\sum_i w^{ii} \geqslant n\det(w_{ij})^{-\frac{1}{n}}=n\left(u^{p-1}f\right)^{-\frac{1}{n}}$$

with (2.7) and the fact $u_1 \geqslant C_3 u^{\frac{p}{n}}$, give

$$\frac{f_1}{fu_1} + \frac{\alpha}{2}\sum_i w^{ii} \geqslant -\frac{|\nabla f|}{fu_1} + \frac{\alpha}{2} n\big(u^{p-1} f\big)^{-\frac{1}{n}} > 0. \tag{2.10}$$

On the other hand, by (2.8) we have

$$\frac{\alpha}{2}\sum_i w^{ii} - \frac{2}{u} \geqslant \frac{\alpha}{2} n\big(u^{p-1} f\big)^{-\frac{1}{n}} - \frac{2}{u} > 0. \tag{2.11}$$

As a result, (2.10) and (2.11) yield (2.9).

Subcase (2_B): $u_1 \leqslant C_3 u^{\frac{p-1}{n}}$, then

$$\frac{u_1}{u^\alpha} \leqslant C_3 \frac{u^{1-\alpha}}{u^{1-\frac{p-1}{n}}} \leqslant \frac{C_3}{M^{1-\frac{p-1}{n}}} u^{1-\alpha}, \tag{2.12}$$

so we get

$$\frac{|\nabla u|}{u^\alpha} \leqslant \frac{C_3}{M^{1-\frac{p-1}{n}}} u^{1-\alpha}, \quad \text{for any } x \in \mathbb{S}^n. \tag{2.13}$$

Let Γ be a great circle connecting x_1 and x_2 with length less than or equal to π, where $u(x_1) = \min u$ and $u(x_2) = \max u$. Then,

$$u(x_2)^{1-\alpha} - u(x_1)^{1-\alpha} = \int_\Gamma du^{1-\alpha} \leqslant (1-\alpha)\int_\Gamma \frac{|\nabla u|}{u^\alpha} \leqslant \frac{C_3(1-\alpha)\pi}{M^{1-\frac{p-1}{n}}} u(x_2)^{1-\alpha},$$

which implies the upper bound of u provided that

$$\frac{C_3(1-\alpha)\pi}{M^{1-\frac{p-1}{n}}} \leqslant \frac{1}{2}.$$

So we get our desired upper bound for $\frac{u_1}{u^\alpha}$ from (2.13). □

In fact Lemma 6.1 of Chou and Wang [11] contains similar gradient estimates near the point of $u = 0$. However that result cannot imply the global upper bounds of u. On the other hand, the $\alpha = \frac{1}{2}$ in (2.1) is the key point for making $C^{1,1}$ estimates of u in Chou and Wang [11] and Guan and Lin [17], which can be realized if p is large enough, such as $p \in (\frac{n+2}{2}, n+1)$. This is the reason why Chou and Wang [11] and Guan and Lin [17] can obtain the $C^{1,1}$ estimates of u only for those p.

The upper bound of u is a corollary of Lemma 2.1.

Corollary 2.2. *Assume that u is an admissible solution to Eq. (1.1) for $0 < p < n+1$. Then there exists a bounded constant $C = C(f, n, p)$ such that*

$$\sup_{\mathbb{S}^n} u \leqslant C. \tag{2.14}$$

Chou and Wang [11] obtained the above theorem for $1 < p < n+1$ by using isoperimetric inequalities, and their method heavily depends on $p > 1$. However, our method is different and uses gradient estimates.

Proof of Corollary 2.2. Let Γ be a great circle connecting x_1 and x_2 with length less than or equal to π, where $u(x_1) = \min u$ and $u(x_2) = \max u$. Then by (2.1) in Lemma 2.1,

$$u(x_2)^{1-\alpha} - u(x_1)^{1-\alpha} = \int_\Gamma du^{1-\alpha} \leqslant (1-\alpha) \int_\Gamma \frac{|\nabla u|}{u^\alpha} \leqslant C. \tag{2.15}$$

On the other hand, from Eq. (1.1), we have at $u(x_1) = \min u$, $u_{ij} \geqslant 0$, and

$$u(x_1) = \min u \leqslant (\max f)^{\frac{1}{n-p+1}}. \tag{2.16}$$

Thus

$$u(x_2) \leqslant C. \qquad \square$$

At the end of this section, we directly obtain a C^1-estimate from Lemma 2.1 and Corollary 2.2.

Corollary 2.3. *If u is an admissible solution to Eq.* (1.1) *for $0 < p < n+1$, then*

$$\max_{\mathbb{S}^n} |\nabla u| \leqslant C(f, n, p). \tag{2.17}$$

3. Detailed analysis of Lemma 2.1 and lower bounds of u

Let Γ be a great circle connecting x_1 and x_2 with length less than or equal to π, where $u(x_1) = \min u$ and $u(x_2) = \max u$. Then

$$u(x_2)^{1-\alpha} - u(x_1)^{1-\alpha} = \int_\Gamma du^{1-\alpha} \leqslant (1-\alpha) \int_\Gamma \frac{|\nabla u|}{u^\alpha} \leqslant C(1-\alpha)\pi, \tag{3.1}$$

provided

$$\frac{|\nabla u|}{u^\alpha} \leqslant C. \tag{3.2}$$

Thus,

$$\max u \leqslant C_4.$$

On the other hand, from Eq. (1.1) we have

$$u(x_1) = \min u \leqslant (\max f)^{\frac{1}{n-p+1}}, \tag{3.3}$$

$$u(x_2) = \max u \geqslant (\min f)^{\frac{1}{n-p+1}}. \tag{3.4}$$

Combining (3.1) with (3.4)

$$u(x_1)^{1-\alpha} = u(x_2)^{1-\alpha} - \int_\Gamma du^{1-\alpha} \geqslant (\min f)^{\frac{1-\alpha}{n-p+1}} - C(1-\alpha)\pi > 0, \tag{3.5}$$

provided C in (3.2) is small enough (see also [33]). We prove that C in (3.2) is small enough to show that u is positive bound from below.

Theorem 3.1. *For any $2 < p < n+1$, there exist two positive constants C_1, C_2 such that the following inequality holds for any solution u to (1.1),*

$$C_1 \leqslant \min_{\mathbb{S}^n} u \leqslant u \leqslant \max_{\mathbb{S}^n} u \leqslant C_2, \tag{3.6}$$

provided

$$\max_{\mathbb{S}^n} |\nabla \log f| \leqslant \frac{p-2}{\pi}. \tag{3.7}$$

Proof. Suppose $G = \frac{|\nabla u|}{u^\beta}$ (β to be determined) attains its maximum at the point $x_0 \in \mathbb{S}^n$. Without loss of generality, choose normal coordinates at x_0 such that

$$u_1 = |\nabla u|, \tag{3.7a}$$

$$u_i = 0 \quad \text{for } i \geqslant 2. \tag{3.7b}$$

We obtain

$$0 = (\log G)_i = \frac{u_{1i}}{u_1} - \beta \frac{u_i}{u},$$

this does imply

$$u_{11} = \beta \frac{u_1^2}{u}, \qquad u_{1i} = 0 \quad \text{for } i \geqslant 2. \tag{3.8}$$

And at x_0,

$$0 \geqslant w^{ij} (\log G)_{ij}$$

states

$$\begin{aligned} 0 &\geqslant \frac{1}{u_1} w^{ij} w_{ij1} - w^{11} + \left(\beta - \beta^2\right) w^{11} \frac{u_1^2}{u^2} - \frac{\beta n}{u} + \beta \sum_i w^{ii} \\ &= \frac{p-1}{u} + \frac{f_1}{f u_1} - \frac{u}{\beta u_1^2 + u^2} + \left(\beta - \beta^2\right) \frac{u_1^2}{\beta u u_1^2 + u^3} - \frac{\beta n}{u} + \beta \sum_i w^{ii}. \end{aligned} \tag{3.9}$$

By (2.14), $u(x_0) < \max_{\mathbb{S}^n} u \leqslant C$. Then we divide u_1 into two cases to deal with the above inequality. Subcase (1_A): $u_1 \leqslant \frac{1}{\pi} u$, then $\frac{u_1}{u^\beta} \leqslant \frac{1}{\pi} u^{1-\beta}$, which states

$$\min_{\mathbb{S}^n} u^{1-\beta} = \max_{\mathbb{S}^n} u^{1-\beta} - \int_\Gamma du^{1-\beta} \geqslant \beta \max_{\mathbb{S}^n} u^{1-\beta} > 0. \tag{3.10}$$

Subcase (1_B): $u_1 > \frac{u}{\pi}$, then $\frac{u^2}{\beta u_1^2 + u^2} = \frac{1}{\beta (\frac{u_1}{u})^2 + 1} < \frac{\pi^2}{\pi^2 + \beta}$, and from (3.9) we have,

$$\begin{aligned}
0 &\geqslant \frac{p-1}{u} + \frac{f_1}{f u_1} - \frac{u}{\beta u_1^2 + u^2} + \left(\beta - \beta^2\right)\frac{u_1^2}{\beta u u_1^2 + u^3} - \frac{\beta n}{u} + \beta \sum_i w^{ii} \\
&= \frac{p-1}{u} + \frac{f_1}{f u_1} + \frac{1}{u}\left[1 - \beta - \frac{(2-\beta)u^2}{\beta u_1^2 + u^2}\right] - \frac{\beta n}{u} + \beta \sum_i w^{ii} \\
&\geqslant \frac{1}{u}\left[p - 2 - \beta n + \frac{(2-\beta)\beta}{\beta + \pi^2}\right] + \frac{f_1}{f u_1}. \qquad (3.11)
\end{aligned}$$

If $p - 2 - \beta n \geqslant 0$, then

$$\frac{1}{\pi} < \frac{u_1}{u} \leqslant \frac{\max_{\mathbb{S}^n} |\nabla \log f|}{p - 2 - n\beta + \frac{(2-\beta)\beta}{\beta+\pi^2}}. \qquad (3.12)$$

This is a contradiction provided

$$\max_{\mathbb{S}^n} |\nabla \log f| \leqslant \frac{p - 2 - n\beta}{\pi}. \qquad (3.13)$$

Thus we have shown that there are positive constants C_1, C_2, such that (3.6) holds. □

Proof of Theorem 1.1. The proof of the theorem is based on the method of continuity. It is equivalent to the a priori estimates because of the openness analysis of Lutwak and Oliker [25]. With the C^0 estimates (3.6) and Corollary 2.3, the method of Guan and Lin [17] shows that the solution of (1.1) is smooth. It suffices to establish the C^2 estimates of u to prove Theorem 1.1 because the C^2 estimates imply that Eq. (1.1) is uniformly elliptic and then the higher regularities of u follow from the Evans–Krylov–Safanov theorem [16]. The C^2 estimates of u to (1.1) follow from Chou and Wang [11] or Guan and Lin [17]. We give the proof here for completeness.

Proposition 3.2. *Assume $f \in C^2(\mathbb{S}^n)$ is a positive function. If u is an admissible solution to Eq. (1.1) with C^0 estimates (3.6) and gradient bound, then for any $p \in \mathbb{R}$, there is a positive constant C depending only on $p, n, \|f\|_{C^2(\mathbb{S}^n)}$ and $\min_{\mathbb{S}^n} f$ such that*

$$\|u\|_{C^2(\mathbb{S}^n)} \leqslant C, \qquad (3.14)$$

and

$$\left\|\frac{1}{u}\right\|_{C^2(\mathbb{S}^n)} \leqslant C. \qquad (3.15)$$

Proof. Here we follow Guan and Lin [17]. It's sufficient to obtain an upper bound for $H = \Delta u + nu$ to get (3.14), since $w_{ij} = (u_{ij} + \delta_{ij} u)$ is semipositive definite. Let

$$H(x_0) = \max_{\mathbb{S}^n} H,$$

and assume matrix (u_{ij}) is diagonal at x_0.

We begin with the following formula for commuting covariant derivatives:

$$\nabla_{ii}\nabla_{jj} = \nabla_{jj}\nabla_{ii} + 2\nabla_{jj} - 2\nabla_{ii}, \qquad (3.16a)$$

$$\nabla_{ii}\Delta = \Delta\nabla_{ii} + 2\Delta - 2n\nabla_{ii}. \qquad (3.16b)$$

Thus

$$\nabla_i H = 0 \quad \text{at } x_0,$$

and

$$0 \geqslant w^{ij} H_{ij} = w^{ij} \Delta w_{ij} + H \sum_i w^{ii} + n^2 \quad \text{at } x_0. \tag{3.17}$$

Applying Δ to the following equation,

$$\det(u_{ij} + u\delta_{ij})^{\frac{1}{n}} = u^{\frac{p-1}{n}} f^{\frac{1}{n}},$$

and by the concavity of $\det^{\frac{1}{n}}$ we have at x_0,

$$\begin{aligned} w^{ij} \Delta w_{ij} &\geqslant \frac{n\Delta u^{\frac{p-1}{n}} f^{\frac{1}{n}}}{u^{\frac{p-1}{n}} f^{\frac{1}{n}}} \\ &\geqslant -C(H+1). \end{aligned} \tag{3.18}$$

On the other hand, if we make order at x_0 such that

$$0 < w_{11} \leqslant w_{22} \leqslant \cdots \leqslant w_{nn},$$

then we get

$$\begin{aligned} \sum_{i=1}^{n} w^{ii} \geqslant \sum_{i=1}^{n-1} w^{ii} &\geqslant (n-1)\left(\prod_{i=1}^{n-1} w^{ii}\right)^{\frac{1}{n-1}} \\ &= (n-1)\frac{(\prod_{i=1}^{n} w^{ii})^{\frac{1}{n-1}}}{(w^{nn})^{\frac{1}{n-1}}} \\ &= (n-1)\big(f u^{p-1}\big)^{-\frac{1}{n-1}} (w_{nn})^{\frac{1}{n-1}} \\ &\geqslant C H^{\frac{1}{n-1}}, \end{aligned} \tag{3.19}$$

for some constant $C > 0$ since u and f are bounded from below and above.

Combining (3.17) and (3.18) with (3.19), there are two positive constants C_1, C_2 such that

$$C_1 H^{\frac{n}{n-1}} - C_2(H+1) \leqslant 0,$$

we obtain the desired bound for H. $\square$

4. An a priori estimates for $p < 2$

In the last section, we make an attempt to give some a priori estimates of the solution to Eq. (1.1) for $p < 2$. Yet, much of the problem still presents a real challenge [9]. Only when $n = 1$, Andrews [4] settled the problem for all real p when f is constant; Gage and Li [14,15] studied the case $p = 0$ with smooth data. Chen [9] and Umanskiy [33] also studied the case $n = 1$ for general p by using a variational method. More recently, Böröczky, Lutwark, Yang and Zhang [5] made great progress for $p = 0$, which gives a necessary and sufficient condition for the prescribed cone-volume measure problem.

Theorem 4.1. *If $0 < p < n + 1$, $0 < \alpha < 1$ and for any positive function $f \in C^{k+2,\alpha}(\mathbb{S}^n)$, assume K has the origin as its center of symmetry. Then for any solution to (1.1), one has*

$$\|u\|_{C^{k+2,\alpha}(\mathbb{S}^n)} \leqslant C\|f\|_{C^{k,1}(\mathbb{S}^n)}, \tag{4.1}$$

where the constant C depends only on n, k, α, $\min_{\mathbb{S}^n} f$ and $\max_{\mathbb{S}^n} f$.

Remark 4.2. In Theorem 4.1, we have assumed that K has the origin as a center of symmetry. If f is an even function, Lutwak [23] showed that K must be centered for $p > 1$. So an interesting problem for Eq. (1.1) is whether it is true for $p < 1$, even for the special case $p = 0$ and $f = 1$ (see [13,23,25]). One can see the latest progress in Andrews [3,4] for dimensions 1 and 2.

According to Corollary (2.2) and Proposition 3.2, we only need the positive lower bound of u to prove Theorem 4.1.

Lemma 4.3. *There exists a positive constant C which depends on α, n, f such that*

$$V(K) \geqslant C. \tag{4.2}$$

In fact, this lemma can be proved by using Proposition 1.3 of Chou and Wang [11] with u's upper bounds. Here we give an alternative proof.

Proof of Lemma 4.3. Let $u(x_1) = \max u$. Then by (2.17),

$$u(x) - u(x_1) = \int_\Gamma du \geqslant -\int_\Gamma |\nabla u| \geqslant -C\,dist(x_1, x), \tag{4.3}$$

where Γ is the minimal geodesic from x_1 to x parametrized by arc-length.

On the other hand, from Eq. (1.1), we have $u(x_1) = \max u$, $u_{ij} \leqslant 0$, and

$$u(x_1) = \max u \geqslant (\min f)^{\frac{1}{n-p+1}}. \tag{4.4}$$

Thus

$$u(x) \geqslant (\min f)^{\frac{1}{n-p+1}} - C\,dist(x_1, x). \tag{4.5}$$

Now we set

$$T = \left\{ x \in \mathbb{S}^n \;\middle|\; u(x) \geqslant \frac{1}{2}(\min f)^{\frac{1}{n-p+1}} \right\},$$

so we have

$$
\begin{aligned}
V(K) &= \frac{1}{n+1} \int_{\mathbb{S}^n} u \det(u_{ij} + u\delta_{ij})\, ds \\
&= \frac{1}{n+1} \int_{\mathbb{S}^n} u^p f\, ds \\
&\geqslant \frac{(\min f)^{\frac{n-p+2}{n-p+1}}}{2^p(n+1)} \int_T ds \\
&\geqslant C. \qquad \square
\end{aligned}
\tag{4.6}
$$

Lemma 4.4 *(John's lemma). (See [21].) Let K be a convex (bound) body in $\mathbb{R}^n$. Then there is an ellipsoid E (after a proper translation) with*

$$\frac{1}{n+1} E \subset K \subset E.$$

With Lemma 4.3, if we can use the above John's lemma to convex body K, then there exists an ellipsoid E such that $\frac{1}{n+1}E \subset K \subset E$, and $V(E) \geqslant C$. From $V(E) \geqslant C$ and u be even, we conclude that there is a sphere B_r^n with radius $r > 0$ such that $B_r^n \subset K$. By the definition of the support function we know that $u(x) \geqslant r > 0$ for any $x \in \mathbb{S}^n$.

Thus, we have proved the following proposition

Proposition 4.5. *Assume $f \in C^1(\mathbb{S}^n)$ is a positive function. If u is an even solution to Eq. (1.1), then u has a lower bound which only depends on p, n, and $\min_{\mathbb{S}^n} f$.*

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